The Neoclassical Growth Model

and Ricardian Equivalence

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Contents

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1. Introduction

This note presents the neoclassical growth model in discrete time. The model is based on microfoundations, which means that the objectives of the economic agents are formulated explicitly, and that their behavior is derived by assuming that they always try to achieve their objectives as well as they can: employment and investment decisions by the firms are derived by assuming that firms maximize profits; consumption and saving decisions by the households are derived by assuming that households maximize their utility.[1]

The model was first developed by Frank Ramsey (Ramsey, 1928). However, while Ramsey's model is in continuous time, the model in this article is presented in discrete time.[2] Furthermore, we do not consider population growth, to keep the presentation as simple as possible.

The set-up of the model is given in section 2. Section 3 derives the model's steady state. The model is then used in section 4 to illustrate *Ricardian equivalence*. Ricardian equivalence is the phenomenon that - given certain assumptions - it turns out to be irrelevant whether the government finances its expenditures by issuing public debt or by raising taxes. Section 5 concludes.

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2. The neoclassical growth model

The representative firm Assume that the production side of the economy is represented by a representative firm, which produces output according to a Cobb-Douglas production function:

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha} \qquad \text{with } 0 < \alpha < 1 \tag{1}$$

Y is aggregate output, K is the aggregate capital stock, L is aggregate labor supply, A is the technology parameter, and the subscript t denotes the time period. The technology parameter A grows at the rate of technological progress g. Labor becomes therefore ever more effective.[3]

The aggregate capital stock depends on aggregate investment I and the depreciation rate δ:

$$K_{t+1} = (1-\delta)K_t + I_t \qquad \text{with } 0 \leq \delta \leq 1 \tag{2}$$

The goods market always clears, such that the firm always sells its total production. Y_t is therefore also equal to the firm's real revenues in period t. The dividends which the firm pays to its shareholders in period t, D_t, are equal to the firm's revenues in period t *minus* its wage expenditures $w_t L_t$ and investment I_t:

$$D_t = Y_t - w_t L_t - I_t \tag{3}$$

where w_t is the real wage in period t. The value of the firm in period t, V_t, is then equal to the present discounted value of the firm's current and future dividends:

$$V_t = \sum_{s=t}^{\infty} \left(\prod_{s'=t+1}^{s} \frac{1}{1+r_{s'}} \right) D_s \tag{4}$$

where $r_{s'}$ is the real rate of return in period s'.

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Taking current and future factor prices as given, the firm hires labor and invests in its capital stock to maximize its current value V_t. This leads to the following first-order-conditions:[4]

$$(1-\alpha)\frac{Y_t}{L_t} = w_t \qquad (5)$$

$$\alpha\frac{Y_{t+1}}{K_{t+1}} = r_{t+1} + \delta \qquad (6)$$

Or in words: the firm hires labor until the marginal product of labor is equal to the marginal cost of labor (which is the real wage w); and the firm invests in its capital stock until the marginal product of capital is equal to the marginal cost of capital (which is the real rate of return r *plus* the depreciation rate δ).

Now substitute the first-order conditions (5) and (6) and the law of motion (2) in the dividend expression (3), and then substitute the resulting equation in the value function (4). This yields the value of the representative firm in the beginning of period t as a function of the initial capital stock and the real rate of return:[5]

$$V_t = K_t(1+r_t) \qquad (7)$$

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The government Every period s, the government has to finance its outstanding public debt B_s, the interest payments on its debt, $B_s r_s$, and government spending G_s. The government can do this by issuing public debt or by raising taxes T_s.[6] Its dynamic budget constraint is therefore given by:

$$B_{s+1} \quad = \quad B_s(1 + r_s) + G_s - T_s \tag{8}$$

where B_{s+1} is the public debt issued in period s (and therefore outstanding in period $s + 1$).

It is natural to require that the government's public debt (or public wealth, if its debt is negative) does not explode over time and become ever larger and larger relative to the size of the economy. Under plausible assumptions, this implies that over an infinitely long horizon the present discounted value of public debt must be zero:

$$\lim_{s\to\infty} \left(\prod_{s'=t}^{s} \frac{1}{1 + r_{s'}} \right) B_{s+1} \quad = \quad 0 \tag{9}$$

This equation is called the *transversality condition*. Combining this transversality condition with the dynamic budget constraint (8) leads to the government's *intertemporal budget constraint*:[7]

$$B_{t+1} \quad = \quad \sum_{s=t+1}^{\infty} \left(\prod_{s'=t+1}^{s} \frac{1}{1 + r_{s'}} \right) T_s - \sum_{s=t+1}^{\infty} \left(\prod_{s'=t+1}^{s} \frac{1}{1 + r_{s'}} \right) G_s \tag{10}$$

Or in words: the public debt issued in period t (and thus outstanding in period $t+1$) must be equal to the present discounted value of future tax revenues *minus* the present discounted value of future government spending. Or also: the public debt issued in period t must be equal to the present discounted value of future primary surpluses.

The representative household Assume that the households in the economy can be represented by a representive household, who derives utility from her current and future consumption:

$$U_t \quad = \quad \sum_{s=t}^{\infty} \left(\frac{1}{1 + \rho} \right)^{s-t} \ln C_s \qquad \text{with } \rho > 0 \tag{11}$$

The parameter ρ is called the subjective discount rate.

Every period s, the household starts off with her assets X_s and receives interest payments $X_s r_s$. She also supplies L units of labor to the representative firm, and

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therefore receives labor income $w_s L$. Tax payments are lump-sum and amount to T_s. She then decides how much she consumes, and how much assets she will hold in her portfolio until period $s+1$. This leads to her *dynamic budget constraint*:

$$X_{s+1} = X_s(1+r_s) + w_s L - T_s - C_s \tag{12}$$

Just as in the case of the government, it is again natural to require that the household's financial wealth (or debt, if her financial wealth is negative) does not explode over time and become ever larger and larger relative to the size of the economy. Under plausible assumptions, this implies that over an infinitely long horizon the present discounted value of the household's assets must be zero:

$$\lim_{s\to\infty} \left(\prod_{s'=t}^{s} \frac{1}{1+r_{s'}} \right) X_{s+1} = 0 \tag{13}$$

Combining this transversality condition with her dynamic budget constraint (12) leads to the household's *intertemporal budget constraint*:[8]

$$\sum_{s=t}^{\infty} \left(\prod_{s'=t+1}^{s} \frac{1}{1+r_{s'}} \right) C_s = X_t(1+r_t) + \sum_{s=t}^{\infty} \left(\prod_{s'=t+1}^{s} \frac{1}{1+r_{s'}} \right) w_s L - \sum_{s=t}^{\infty} \left(\prod_{s'=t+1}^{s} \frac{1}{1+r_{s'}} \right) T_s \tag{14}$$

Or in words: the present discounted value in period t of her current and future consumption must be equal to the value of her assets in period t (including interest payments) *plus* the present discounted value of current and future labor income *minus* the present discounted value of current and future tax payments.

The household takes X_t and the current and future values of r, w, and T as given, and chooses her consumption path to maximize her utility (11) subject to her intertemporal budget constraint (14). This leads to the following first-order condition (which is called the *Euler equation*):

$$C_{s+1} = \frac{1+r_{s+1}}{1+\rho} C_s \tag{15}$$

Combining with the intertemporal budget constraint leads then to the current value of her consumption:

$$C_t = \frac{\rho}{1+\rho} \left\{ X_t(1+r_t) + \sum_{s=t}^{\infty} \left(\prod_{s'=t+1}^{s} \frac{1}{1+r_{s'}} \right) w_s L - \sum_{s=t}^{\infty} \left(\prod_{s'=t+1}^{s} \frac{1}{1+r_{s'}} \right) T_s \right\} \tag{16}$$

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Or in words: every period t, the household consumes a fraction $\rho/(1+\rho)$ of her total wealth, which consists of her financial wealth $X_t(1+r_t)$ and her human wealth (i.e. the present discounted value of her current and future labor income), *minus* the present discounted value of all her current and future tax obligations.[9]

Equilibrium Every period, the factor markets clear. For the labor market, we already implicitly assumed this by using the same notation (L) for the representative household's labor supply and the representative firm's labor demand.

Equilibrium in the capital market requires that the representative household holds all the shares of the representative firm and the outstanding government bonds. The value of the representative firm in the beginning of period $t+1$ is V_{t+1}, such that the total value of the shares which the household can buy at the end of period t is given by $V_{t+1}/(1+r_{t+1})$. The value of the government bonds which the household can buy at the end of period t is equal to the total public debt issued in period t, which is denoted by B_{t+1}. This implies that

$$X_{t+1} = \frac{V_{t+1}}{1+r_{t+1}} + B_{t+1} \tag{17}$$

Equilibrium in the goods market requires that the total production is consumed, invested or purchased by the government, such that $Y_t = C_t + I_t + G_t$. Note that equilibrium in the goods market is automatic if the labor and the capital markets are also in equilibrium (because of Walras' law).

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3. The steady state

It is often useful to analyse how the economy behaves in steady state. To derive the steady state, we need to impose some restrictions on the time path of government spending. Let us therefore assume that government spending G grows at the rate of technological progress g:

$$G_{s+1} = G_s(1+g) \tag{18}$$

To derive the steady state, we start from an educated guess: let us suppose that in the steady state consumption also grows at the rate of technological progress g. We can then derive the values of the other variables, and verify that the model can indeed be solved (such that our educated guess turns out to be correct).

The steady state values of output, capital, investment, consumption, the real wage and the real interest rate are then given by the following expressions:[10]

$$Y_t^* = \left(\frac{\alpha}{r^*+\delta}\right)^{\frac{\alpha}{1-\alpha}} A_t L \tag{19}$$

$$K_t^* = \left(\frac{\alpha}{r^*+\delta}\right)^{\frac{1}{1-\alpha}} A_t L \tag{20}$$

$$I_t^* = (g+\delta)\left(\frac{\alpha}{r^*+\delta}\right)^{\frac{1}{1-\alpha}} A_t L \tag{21}$$

$$C_t^* = \left[1-\alpha\frac{g+\delta}{r^*+\delta}\right]\left(\frac{\alpha}{r^*+\delta}\right)^{\frac{\alpha}{1-\alpha}} A_t L - G_t^* \tag{22}$$

$$w_t^* = (1-\alpha)\left(\frac{\alpha}{r^*+\delta}\right)^{\frac{\alpha}{1-\alpha}} A_t \tag{23}$$

$$r^* = (1+\rho)(1+g)-1 \tag{24}$$

...where a superscript * shows that the variables are evaluated in the steady state. Recall that the technology parameter A and government spending G grow at rate g while labor supply L remains constant over time. It then follows from the equations above that the steady state values of aggregate output Y^*, the aggregate capital stock K^*, aggregate investment I^*, aggregate consumption C^* and the real wage w^* all grow at the rate of technological progress g.

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We can now draw two conclusions:

First, suppose that the government increases government spending G_t^* and afterwards continues to have G^* growing over time at rate g (such that government spending is permanently higher). It then follows from equations (19) until (24) that aggregate consumption decreases one-for-one with the higher government spending. The rest of the economy, however, is not affected: aggregate output, the capital stock, investment, the real wage and the real interest rate do not change as a result of a permanent shock in government spending. So government spending crowds out consumption.

Second, the way how the government finances its spending turns out to be irrelevant for the behavior of the economy: whether the government finances its spending by raising taxes or by issuing public debt, does not matter. This phenomenon is called *Ricardian equivalence*. Ricardian equivalence actually holds not only in steady state, but also outside steady state. In the next section, we explore the reason for Ricardian equivalence in more detail.

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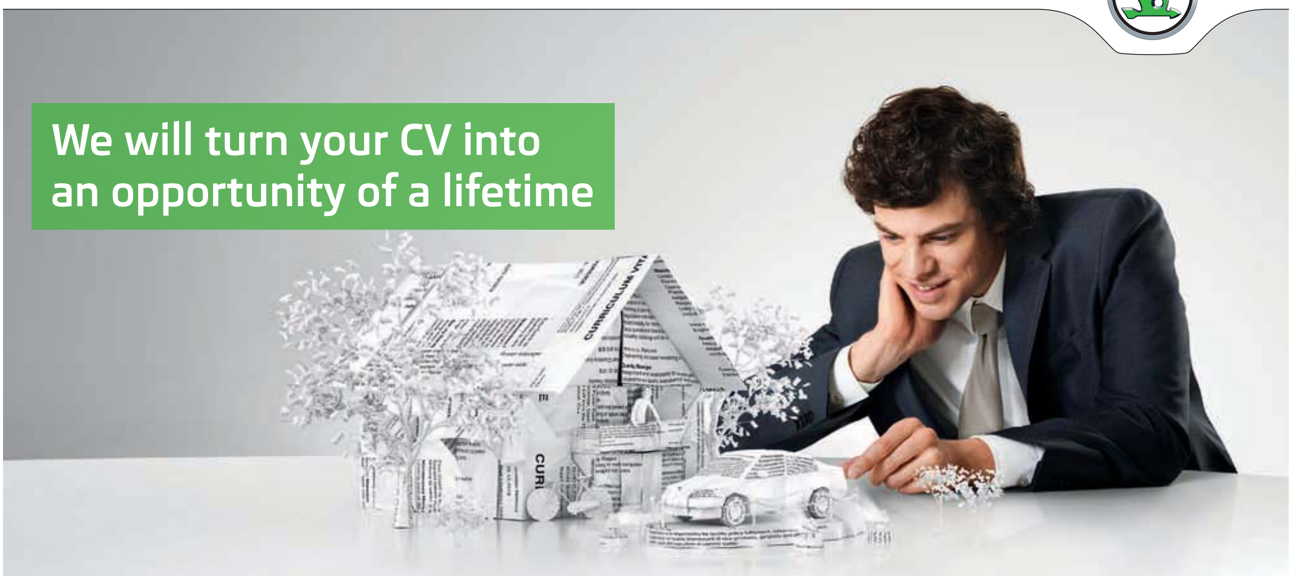

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4. Ricardian equivalence

Let us consider again the intertemporal budget constraint of the representative household, equation (14). Recall that the household's assets consist of the shares of the representative firm and the public debt, such that $X_t = V_t/(1+r_t) + B_t$. Substituting in (14) yields:

$$\begin{aligned}\sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)C_s &= V_t + B_t(1+r_t) + \sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)w_s L \\ &\quad - \sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)T_s \qquad (25)\end{aligned}$$

Now replace B_t by the right-hand-side of the government's budget constraint (equation (10), but moved backwards with one period):

$$\begin{aligned}\sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)C_s &= V_t + \sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)T_s - \sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)G_s \\ &\quad + \sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)w_s L - \sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)T_s \qquad (26)\end{aligned}$$

The present discounted value of tax payments then cancels out, such that the household's intertemporal budget constraint becomes:

$$\sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)C_s = V_t + \sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)w_s L - \sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)G_s \qquad (27)$$

This is a crucial result. It means that from the household's point of view, only the present discounted value of government spending matters. The precise time path of tax payments and the size of the public debt are irrelevant. The reason for this is that every increase in public debt must sooner or later be matched by an increase in taxes. Households therefore do not consider their government bonds as net wealth, because they realize that sooner or later they will have to pay taxes to the government such that the government can retire the bonds. From the household's point of view, it is therefore irrelevant whether the government has a large public debt or not: in the first case, households will have a lot of assets, but expect to pay a lot of taxes later on; in the second case, households will have fewer assets, but feel compensated for that as they also anticipate lower taxes.

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5. Conclusions

This note presented the neoclassical growth model, and solved for the steady state. In the neoclassical growth model, it is irrelevant whether the government finances its expenditures by issuing debt or by raising taxes. This phenomenon is called Ricardian Equivalence.

Of course, the real world is very different from the neoclassical growth model. Consequently, there are many reasons why Ricardian Equivalence may not hold in reality. But nevertheless, the neoclassical growth model is a useful starting point for more complicated dynamic general equilibrium models, and the principle of Ricardian Equivalence often serves as a benchmark to evaluate the effect of government debt in more realistic settings.

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[1] Note that the Solow growth model (Solow, 1956) is sometimes called the neoclassical growth model as well. But the Solow model is *not* based on microfoundations, as it assumes an exogenous saving rate.

[2] The stochastic growth model, which is at the heart of modern macroeconomic research, is in essence a stochastic version of the neoclassical growth model, and is usually presented in discrete time as well.

[3] This type of technological progress is called *labor-augmenting* or *Harrod-neutral* technological progress.

[4] See appendix A for derivations.

[5] See appendix A for a derivation.

[6] In general, there is a third source of revenue for the government, namely seigniorage income. For simplicity, we make abstraction from this, and assume that seigniorage income is zero.

[7] See appendix A for derivations.

[8] The derivation is similar as for the government's intertemporal budget constraint, and is given in appendix A.

[9] Derivations of equations (15) and (16) are given in appendix A.

[10] Derivations are given in appendix B.

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Appendix A

A1. The maximization problem of the representative firm

The maximization problem of the firm can be rewritten as:

$$V_t(K_t) = \max_{\{L_t, I_t\}} \left\{ Y_t - w_t L_t - I_t + \frac{1}{1+r_{t+1}} V_{t+1}(K_{t+1}) \right\} \quad (A.1)$$

$$s.t. \quad Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$$

$$K_{t+1} = (1-\delta)K_t + I_t$$

The first-order conditions for L_t, respectively I_t, are:

$$(1-\alpha)K_t^{\alpha} A_t^{1-\alpha} L_t^{-\alpha} - w_t = 0 \quad (A.2)$$

$$-1 + \frac{1}{1+r_{t+1}} \frac{\partial V_{t+1}(K_{t+1})}{\partial K_{t+1}} = 0 \quad (A.3)$$

In addition, the envelope theorem implies that

$$\frac{\partial V_t(K_t)}{\partial K_t} = \alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha} + \frac{1}{1+r_{t+1}} \frac{\partial V_{t+1}(K_{t+1})}{\partial K_{t+1}}(1-\delta) \quad (A.4)$$

Substituting the production function in (A.2) gives equation (5):

$$(1-\alpha)\frac{Y_t}{L_t} = w_t$$

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Substituting (A.3) in (A.4) yields:

$$\frac{\partial V_t(K_t)}{\partial K_t} = \alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha} + (1-\delta)$$

Moving one period forward, and substituting again in (A.3) gives:

$$-1 + \frac{1}{1+r_{t+1}}\left[\alpha K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha} + (1-\delta)\right] = 0$$

Reshuffling leads to:

$$1 + r_{t+1} = \alpha K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha} + (1-\delta)$$

Substituting the production function in the equation above gives then equation (6):

$$\alpha\frac{Y_{t+1}}{K_{t+1}} = r_{t+1} + \delta$$

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A2. The equilibrium value of the representative firm

First substitute the first-order conditions (5) and (6) and the law of motion (2) in the dividend expression (3):

$$\begin{aligned} D_t &= Y_t - (1-\alpha)Y_t - I_t \\ &= \alpha Y_t - I_t \\ &= K_t(r_t+\delta) - [K_{t+1} - (1-\delta)K_t] \\ &= K_t(1+r_t) - K_{t+1} \end{aligned}$$

Substituting in the value function (4) gives then the equilibrium value of V_t, equation (7):

$$\begin{aligned} V_t &= K_t(1+r_t) - K_{t+1} + \frac{K_{t+1}(1+r_{t+1}) - K_{t+2}}{1+r_{t+1}} + \frac{K_{t+2}(1+r_{t+2}) - K_{t+3}}{(1+r_{t+1})(1+r_{t+2})} + \cdots \\ &= K_t(1+r_t) \end{aligned}$$

A3. The goverment's intertemporal budget constraint

Let us first rewrite the government's dynamic budget constraint (8):

$$B_t = \frac{B_{t+1}}{1+r_t} + \frac{T_t}{1+r_t} - \frac{G_t}{1+r_t} \tag{A.5}$$

Moving this equation one period forward gives an expression for B_{t+1}, which we can use to eliminate B_{t+1} in the equation above:

$$\begin{aligned} B_t &= \frac{1}{1+r_t}\left[\frac{B_{t+2}}{1+r_{t+1}} + \frac{T_{t+1}}{1+r_{t+1}} - \frac{G_{t+1}}{1+r_{t+1}}\right] + \frac{T_t}{1+r_t} - \frac{G_t}{1+r_t} \\ &= \frac{B_{t+2}}{(1+r_t)(1+r_{t+1})} + \frac{T_t}{1+r_t} + \frac{T_{t+1}}{(1+r_t)(1+r_{t+1})} \\ &\quad - \frac{G_t}{1+r_t} - \frac{G_{t+1}}{(1+r_t)(1+r_{t+1})} \end{aligned} \tag{A.6}$$

We can then move equation (A.5) two periods forward to obtain an expression for B_{t+2}, and replace B_{t+2} in (A.6). Continuing in this manner eventually leads to

$$B_t = \lim_{s\to\infty}\left(\prod_{s'=t}^{s}\frac{1}{1+r_{s'}}\right)B_{s+1} + \sum_{s=t}^{\infty}\left(\prod_{s'=t}^{s}\frac{1}{1+r_{s'}}\right)T_s - \sum_{s=t}^{\infty}\left(\prod_{s'=t}^{s}\frac{1}{1+r_{s'}}\right)G_s$$

Using the transversality condition (9) leads then to the government's intertemporal budget constraint:

$$B_t(1+r_t) = \sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)T_s - \sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)G_s$$

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...which can be rewritten as in equation (10):

$$B_{t+1} = \sum_{s=t+1}^{\infty} \left(\prod_{s'=t+1}^{s} \frac{1}{1+r_{s'}} \right) T_s - \sum_{s=t+1}^{\infty} \left(\prod_{s'=t+1}^{s} \frac{1}{1+r_{s'}} \right) G_s$$

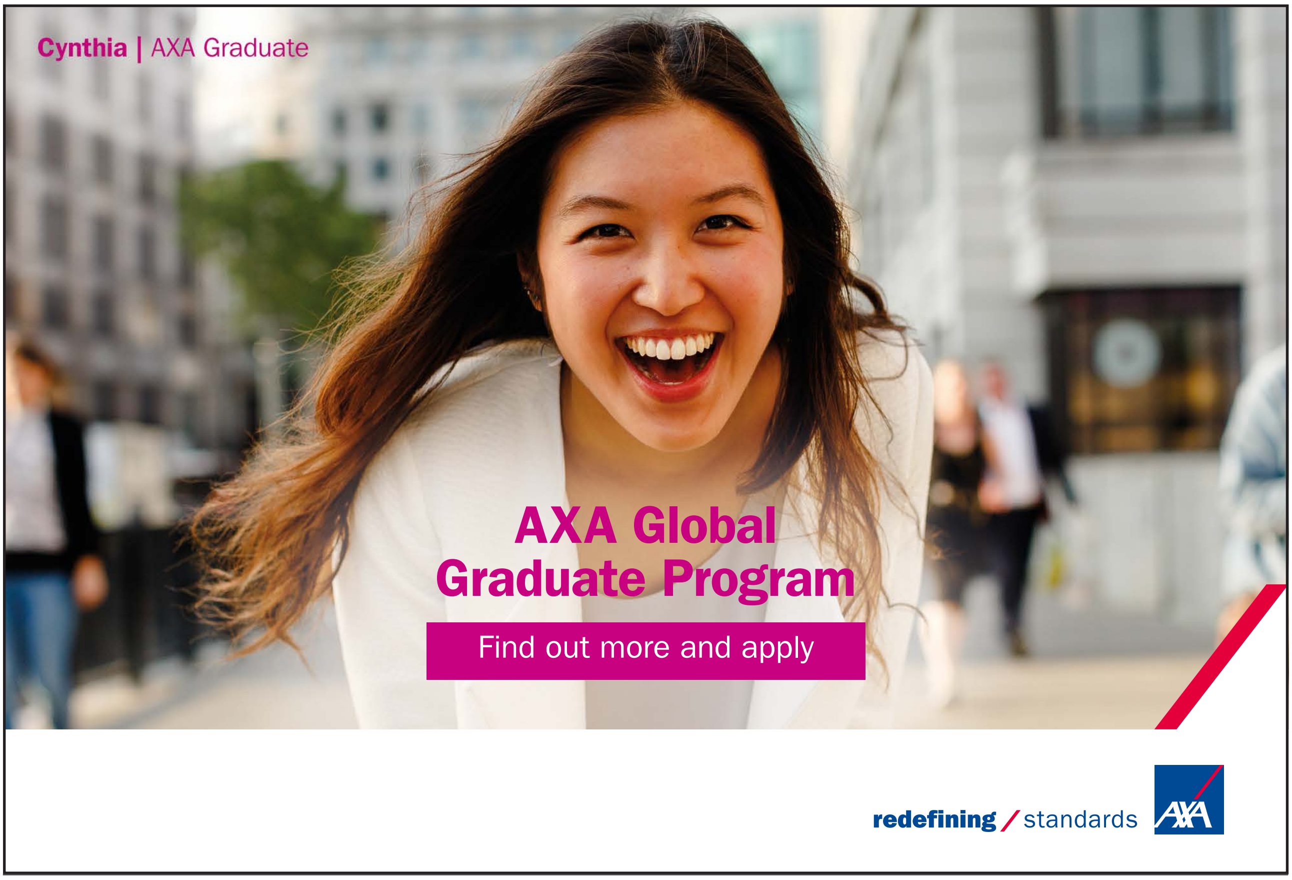
A4. The representative household's intertemporal budget constraint

Let us first rewrite the representative household's dynamic budget constraint (12):

$$X_t = \frac{X_{t+1}}{1+r_t} - \frac{w_t L}{1+r_t} + \frac{T_t}{1+r_t} + \frac{C_t}{1+r_t} \tag{A.7}$$

Moving this equation one period forward gives an expression for X_{t+1}, which we can use to eliminate X_{t+1} in the equation above:

$$\begin{aligned} X_t &= \frac{1}{1+r_t}\left[\frac{X_{t+2}}{1+r_{t+1}} - \frac{w_{t+1}L}{1+r_{t+1}} + \frac{T_{t+1}}{1+r_{t+1}} + \frac{C_{t+1}}{1+r_{t+1}}\right] - \frac{w_t L}{1+r_t} + \frac{T_t}{1+r_t} \\ &\quad + \frac{C_t}{1+r_t} \\ &= \frac{X_{t+2}}{(1+r_t)(1+r_{t+1})} - \frac{w_t L}{1+r_t} - \frac{w_{t+1}L}{(1+r_t)(1+r_{t+1})} + \frac{T_t}{1+r_t} + \frac{T_{t+1}}{(1+r_t)(1+r_{t+1})} \\ &\quad + \frac{C_t}{1+r_t} + \frac{C_{t+1}}{(1+r_t)(1+r_{t+1})} \end{aligned} \tag{A.8}$$



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We can then move equation (A.7) two periods forward to obtain an expression for X_{t+2}, and replace X_{t+2} in (A.8). Continuing in this manner eventually leads to

$$\begin{aligned} X_t &= \lim_{s\to\infty}\left(\prod_{s'=t}^{s}\frac{1}{1+r_{s'}}\right)X_{s+1}-\sum_{s=t}^{\infty}\left(\prod_{s'=t}^{s}\frac{1}{1+r_{s'}}\right)w_sL \\ &\quad+\sum_{s=t}^{\infty}\left(\prod_{s'=t}^{s}\frac{1}{1+r_{s'}}\right)T_s+\sum_{s=t}^{\infty}\left(\prod_{s'=t}^{s}\frac{1}{1+r_{s'}}\right)C_s \end{aligned}$$

Using the transversality condition (13) and rearranging, leads then to the household's intertemporal budget constraint:

$$\sum_{s=t}^{\infty}\left(\prod_{s'=t}^{s}\frac{1}{1+r_{s'}}\right)C_s = X_t+\sum_{s=t}^{\infty}\left(\prod_{s'=t}^{s}\frac{1}{1+r_{s'}}\right)w_sL-\sum_{s=t}^{\infty}\left(\prod_{s'=t}^{s}\frac{1}{1+r_{s'}}\right)T_s$$

...which can be rewritten as in equation (14):

$$\begin{aligned} \sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)C_s &= X_t(1+r_t)+\sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)w_sL \\ &\quad-\sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)T_s \end{aligned}$$

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A5. The maximization problem of the representative household

The maximization problem of the household can be rewritten as:

$$U_t(X_t) = \max_{\{C_t\}} \left\{ \ln C_t + \frac{1}{1+\rho} U_{t+1}(X_{t+1}) \right\} \tag{A.9}$$

$$s.t. \quad X_{t+1} = X_t(1+r_t) + w_t L - T_t - C_t$$

The first-order condition for C_t is:

$$\frac{1}{C_t} - \frac{1}{1+\rho}\frac{\partial U_{t+1}(X_{t+1})}{\partial X_{t+1}} = 0 \tag{A.10}$$

In addition, the envelope theorem implies that

$$\frac{\partial U_t(X_t)}{\partial X_t} = \frac{1}{1+\rho}\frac{\partial U_{t+1}(X_{t+1})}{\partial X_{t+1}}(1+r_t) \tag{A.11}$$

Substituting (A.10) in (A.11) yields:

$$\frac{\partial U_t(X_t)}{\partial X_t} = (1+r_t)\frac{1}{C_t}$$

Moving one period forward, and substituting again in (A.10) gives:

$$\frac{1}{C_t} - \frac{1+r_{t+1}}{1+\rho}\frac{1}{C_{t+1}} = 0$$

Rearranging leads then to the Euler equation (15):

$$C_{t+1} = \frac{1+r_{t+1}}{1+\rho} C_t \tag{A.12}$$

A6. The consumption level of the representative household

First note that repeatedly using the Euler-equation (15) allows us to eliminate all future values of C from the left-hand-side of equation (14):

$$\sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)C_s = \sum_{s=t}^{\infty}\left(\frac{1}{1+\rho}\right)^{s-t} C_t$$

$$= \frac{1+\rho}{\rho} C_t$$

Substituting in (14) and rearranging yields then equation (16):

$$C_t = \frac{\rho}{1+\rho}\left\{X_t(1+r_t) + \sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)w_s L - \sum_{s=t}^{\infty}\left(\prod_{s'=t+1}^{s}\frac{1}{1+r_{s'}}\right)T_s\right\}$$

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Appendix B

If C grows at rate g, the Euler equation (15) implies that

$$C_s^*(1+g) \quad = \quad \frac{1+r^*}{1+\rho} C_s^*$$

Rearranging gives then the gross real rate of return $1+r^*$:

$$1 + r^* = (1+g)(1+\rho)$$

which immediately leads to equation (24).

Subsituting in the firm's first-order condition (6) gives:

$$\alpha \frac{Y_{t+1}^*}{K_{t+1}^*} \quad = \quad r^* + \delta$$

Using the production function (1) to eliminate Y yields:

$$\alpha K_{t+1}^{*\alpha-1}(A_{t+1}L)^{1-\alpha} \quad = \quad r^* + \delta$$

Rearranging gives then the value of K_{t+1}^*:

$$K_{t+1}^* \quad = \quad \left(\frac{\alpha}{r^* + \delta}\right)^{\frac{1}{1-\alpha}} A_{t+1}L$$

which is equivalent to equation (20).

Substituting in the production function (1) gives then equation (19):

$$Y_t^* \quad = \quad \left(\frac{\alpha}{r^* + \delta}\right)^{\frac{\alpha}{1-\alpha}} A_t L$$

Substituting (19) in the first-order condition (5) gives equation (23):

$$w_t^* \quad = \quad (1-\alpha)\left(\frac{\alpha}{r^* + \delta}\right)^{\frac{\alpha}{1-\alpha}} A_t$$

Substituting (20) in the law of motion (2) yields:

$$\left(\frac{\alpha}{r^* + \delta}\right)^{\frac{1}{1-\alpha}} A_{t+1}L \quad = \quad (1-\delta)\left(\frac{\alpha}{r^* + \delta}\right)^{\frac{1}{1-\alpha}} A_t L + I_t^*$$

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such that I_t^* is given by:

$$\begin{aligned} I_t^* &= \left(\frac{\alpha}{r^*+\delta}\right)^{\frac{1}{1-\alpha}} A_{t+1}L - (1-\delta)\left(\frac{\alpha}{r^*+\delta}\right)^{\frac{1}{1-\alpha}} A_tL \\ &= \left(\frac{\alpha}{r^*+\delta}\right)^{\frac{1}{1-\alpha}} \left[(1+g)-(1-\delta)\right] A_tL \\ &= (g+\delta)\left(\frac{\alpha}{r^*+\delta}\right)^{\frac{1}{1-\alpha}} A_tL \end{aligned}$$

...which is equation (21).

Consumption C^* can then be computed from the equilibrium condition in the goods market:

$$\begin{aligned} C_t^* &= Y_t^* - I_t^* - G_t^* \\ &= \left(\frac{\alpha}{r^*+\delta}\right)^{\frac{\alpha}{1-\alpha}} A_tL - (g+\delta)\left(\frac{\alpha}{r^*+\delta}\right)^{\frac{1}{1-\alpha}} A_tL - G_t^* \\ &= \left[1-\alpha\frac{g+\delta}{r^*+\delta}\right]\left(\frac{\alpha}{r^*+\delta}\right)^{\frac{\alpha}{1-\alpha}} A_tL - G_t^* \end{aligned}$$

Now recall that on the balanced growth path, A and G grow at the rate of technological progress g. The equation above then implies that C^* also grows at the rate g, such that our initial educated guess turns out to be correct.

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